Fluid-dynamic loading of pipes conveying fluid with a laminar mean-flow velocity profile

J. Kutin*, I. Bajsić

Laboratory of Measurements in Process Engineering, Faculty of Mechanical Engineering, University of Ljubljana, Aškerčeva 6, SI-1000 Ljubljana, Slovenia

* Corresponding author. Tel.: +386 1 4771 307; fax: +386 1 4771 118.
E-mail address: joze.kutin@fs.uni-lj.si (J. Kutin).

Abstract

The fluid-conveying pipe is a fundamental dynamical problem in the field of fluid-structure interactions. The possibility of modelling such a system analytically is mainly dependent on the availability of suitable analytical models for the fluid-dynamic loading acting on a vibrating pipe. The aim of this paper is an analytical study of the velocity profile effects for a straight pipe. The main contribution is the derived asymptotic model for the fluid-dynamic loading of the laminar and uniform mean flows that is applicable for any circumferential wavenumber of the mode shape of the pipe. The velocity profile effects are expressed in terms of the correction factors for the fluid-dynamic loading, which consists of three components being related to the translational, Coriolis and centrifugal accelerations of the fluid. This model also takes into account the effects of the fluid compressibility and the finite pipe length. The asymptotic model is derived from the solutions of the Pridmore-Brown equation for the Fourier transform of the vibrational fluid pressure. The solution procedure is based on the Frobenius power series method. The results about the velocity profile effects are compared with the previous, accessible studies in this field. As an application case, the discussed model is used to predict the flow effects on the first-mode natural frequency of a beam-type pipe with both ends clamped.

Keywords: Pipe conveying fluid; Fluid-dynamic loading; Analytical model; Velocity profile effect; Uniform mean flow; Laminar mean flow

1. Introduction

The fluid-conveying pipe is a fundamental dynamical problem in the field of fluid-structure interactions (Paidoussis, 1998, 2004). This topic has many direct engineering applications, as well as serving as a model for understanding more complex systems and for searching new dynamic features and phenomena. The use of analytical modelling on fluid-conveying pipes is dependent on the availability of suitable analytical mathematical models for the fluid-dynamic loading acting on a vibrating pipe. The most common models are based on the assumptions of a mean flow with a uniform velocity profile and a vibrational flow characterized by means of the one-dimensional theory or the wave theory. Here, the term ‘mean flow’ defines unperturbed fluid flow in a pipe at rest and the term ‘vibrational flow’ defines flow perturbations relative to the mean flow resulting from the pipe vibration.
The one-dimensional model of the fluid-dynamic force is appropriate for long pipes vibrating in bending, beam-type modes (Païdoussis, 1998). The early contributions to the evolution of the one-dimensional models of fluid-conveying pipes are from Housner (1952), Benjamin (1961), Gregory and Païdoussis (1966), etc. Assuming a straight pipe with a one-dimensional (mean and vibrational) internal fluid flow, the fluid-dynamic force per unit length can be written as:

\[ f_d(x,t) = -M_f \left( \frac{\partial^2 w}{\partial t^2} + 2\bar{V} \frac{\partial^2 w}{\partial t \partial x} + \bar{V}^2 \frac{\partial^2 w}{\partial x^2} \right), \]

where \( w(x,t) \) is the pipe lateral deflection, \( M_f \) is the fluid mass per unit length and \( \bar{V} \) is the fluid average axial velocity. The fluid-dynamic force consists of three components, which are related to the translational acceleration \( \frac{\partial^2 w}{\partial t^2} \), the Coriolis acceleration \( 2\bar{V} \frac{\partial^2 w}{\partial t \partial x} \) and the centrifugal acceleration \( \bar{V}^2 \frac{\partial^2 w}{\partial x^2} \) of the fluid.

The assumption of a uniform vibrational flow field over the pipe cross-section is unsuitable for pipes that are vibrating in higher circumferential, shell-type modes, as well as for relatively short pipes that are vibrating in beam-type modes. In such cases, the analytical mathematical models for the fluid-dynamic loading are usually based on the wave equation, with the travelling or standing wave solutions (Paidoussis, 2004). The early contributions to the evolution of the wave models of fluid-conveying pipes are from Païdoussis and Denise (1972), Weaver and Unny (1973), Weaver and Myklatun (1973), Shayo and Ellen (1974), etc. Following the standing-wave modelling approach, the fluid-dynamic pressure on a straight circular pipe can be written as:

\[ p_d(x,\theta,t) = \bar{p}_d(x) \cos n\theta e^{i\omega t}, \]

\[ \bar{p}_d(x) = \rho_0 R \frac{1}{2\pi} \int_{-\infty}^{\infty} I_n(\lambda^{(c)}R) \left( \omega - \bar{V}\lambda \right)^2 \int_0^L \bar{w}_r(y) e^{i(y-x)} dy d\lambda, \]

where it is assumed that the pipe vibrates periodic circumferentially and harmonic in time with the radial displacement \( w_r(x,\theta,t) = \bar{w}_r(x) \cos n\theta e^{i\omega t} \), the fluid is compressible with a uniform mean-flow velocity profile, \( \lambda^{(c)} = \left[ \lambda^2 - (\omega - \bar{V}\lambda)^2 / \bar{c}^2 \right]^{1/2} \), \( I_n(\lambda^{(c)}R) = R \frac{dI_n(\lambda^{(c)}r)}{dr} \bigg|_{r=R} \), \( I_n \) is the modified Bessel function of the first kind, \( n \) is the circumferential wavenumber, \( \lambda \) is the axial wavenumber, \( \omega \) is the angular frequency, \( \rho_0 \) is the fluid mean-density, \( \bar{c} \) is the speed of sound in the fluid, \( R \) is the internal radius of the pipe and \( L \) is the pipe length (see Fig. 1). Such an expression for the fluid-dynamic loading also shows three components, corresponding to three terms of \( (\omega - \bar{V}\lambda)^2 = \omega^2 - 2\bar{V}\lambda\omega + \bar{V}^2 \lambda^2 \), which can be related to the translational, Coriolis and centrifugal accelerations, respectively. These components become even more explicit under the assumption of an incompressible fluid and a long pipe \( (L^2 >> R^2) \):
\[ p_d(x, \theta, t) = -\frac{\rho_0 R}{n} \left( \frac{\partial^2 w_r}{\partial t^2} + 2V \frac{\partial^2 w_r}{\partial t \partial x} + V^2 \frac{\partial^2 w_r}{\partial x^2} \right). \] (3)

For the beam-type mode \((n = 1)\) the integral effect of the fluid-dynamic pressure \((3)\) in the direction of the pipe vibration equals the one-dimensional model \((1)\).

Fig. 1. Sketch of a pipe conveying fluid and the first three circumferential mode shapes.

The common assumption in all the presented analytical models \((1)\)–\((3)\) is the mean flow with a uniform velocity profile. For a fully developed pipe flow the form of the velocity profile actually depends on the Reynolds number \(Re = \frac{\bar{V}D}{\nu}\) and the pipe roughness, where \(D = 2R\) and \(\nu\) is the fluid kinematic viscosity (Blevins, 1992). The critical Reynolds number, \(Re \approx 2000\), represents the approximate boundary between laminar flow \((Re < 2000)\) and transitional/turbulent flow \((Re > 2000)\). The velocity profile approaches the uniform distribution for turbulent flow at high Reynolds numbers, but the velocity profile may vary considerably at low Reynolds numbers. A fully developed laminar flow in a straight circular pipe (Hagen-Poiseuille flow) has a parabolic velocity profile:

\[ V_s(r) = 2\bar{V} \left( 1 - \frac{r^2}{R^2} \right), \] (4)

with the maximum velocity at the centre of the pipe being twice as large as the average velocity. The laminar mean flow \((Re < 2000)\) is typical for the low flow range of applications with macroscale pipes conveying fluid, whereas it may cover the whole flow range of applications with microscale pipes conveying fluid (see, e.g., Rinaldi et al., 2010; Wang, 2010; Yin et al., 2011; Arani et al., 2012a, 2012b; Wang et al., 2013). Applications with even smaller dimensions represent nano-tubes conveying fluid (see, e.g., Wang and Ni, 2009; Rashidi et al., 2012; Kaviani and Mirdamadi, 2012; Mirramezani et al., 2013). On such a small dimensional scale, where the mean free path of a molecule is comparable to the length scale of the problem (the Knudsen number is
close to or greater than one), the continuum assumption of fluid mechanics may no longer be a good approximation and the no-slip condition (such as in Eq. (4)) breaks down.

There are not many accessible studies that deal with the analytical modelling of the velocity profile effects on the fluid-dynamic loading of fluid-conveying pipes. In the field of Coriolis flow meters, which represent the application of fluid-conveying pipes for measuring the mass flow rate and the density of fluids, the velocity profile effects have been analytically studied using the weight vector theory. The weight vector theory of Coriolis flow meters was set up by Hemp (1988, 1994), by deriving the linearized, three-dimensional, fluid-dynamic body force and further applying the reciprocity principle to estimate its effect on the pipe vibration. This model can be used for any steady mean flow, but within the limits of relatively small effects that are linearly dependent on the mean-flow velocity. Kutin et al. (2005, 2006) and Hemp and Kutin (2006) performed calculations for different cases using the weight vector theory. The results for long pipes conveying fluid ($L^2 \gg R^2$) show negligible velocity profile effects for the beam-type modes, whereas the change of velocity profile considerably decreases the Coriolis flow measuring effect for the shell-type modes (e.g., a correction factor of 2/3 for $n = 2$ and a laminar velocity profile). As shown in Section 3 of the present paper, the predicted effects using the weight vector theory for fully developed flows can be directly related to the correction factor for the Coriolis-acceleration component of the fluid-dynamic loading.

Guo et al. (2010) and Hellum et al. (2010) proposed (at almost the same time) a similar approach for considering the velocity profile effects of the mean flow for long beam-type pipes conveying fluid. The proposed corrections to the fluid-dynamic force are based on a definition of the momentum correction factor. The Coriolis-acceleration component with a linear dependency on the fluid velocity is found not to be affected by the velocity profile. On the other hand, the centrifugal-acceleration component has a quadratic dependency on the fluid velocity and its integral over the pipe cross-section is consequently found to increase for the non-uniform velocity profiles (e.g., the correction factor of 4/3 for the laminar velocity profile). However, the results in the present paper (see Section 3) reveal some doubt about the validity of this approach, because it omits the effects of the secondary flows.

The aim of the present paper is a study of the fluid-dynamic loading of a straight pipe in terms of the effects of the mean flow with a laminar velocity profile. The results are compared with those for a uniform velocity profile. The main contribution of the paper is an approximate, asymptotic model for the fluid-dynamic loading that considers the flow effects up to the second order in the fluid velocity and is applicable for any circumferential wavenumber of the mode shape of a pipe. The asymptotic model is derived from the solutions of the Pridmore-Brown equation for the Fourier transform of the vibrational fluid pressure, which considers the effects of the axisymmetric mean flow. The solution procedure is based on the Frobenius power series method.

The paper is organized as follows. Section 2 summarizes a derivation of the governing equations (Section 2.1), defines the boundary conditions on the surface of the vibrating pipe (Section 2.2), and describes the solution procedure (Section 2.3). Section
3 presents the asymptotic model for the fluid-dynamic model and compares the findings with those already published. Section 4 presents the application of the discussed model for the fluid-dynamic loading on a pipe that is clamped at both ends and vibrating in the beam-type mode.

2. Mathematical model for the fluid-dynamic loading

2.1. Governing equations

The derivation of the mathematical model proceeds from the so-called hydrodynamic/acoustic splitting method. This approach is often employed in computational methods for compressible flow at low Mach numbers to increase the computational efficiency (see, e.g., Shen and Sorensen, 1999; Munz et al., 2007). The fluid flow is split into the incompressible mean flow part, which refers to the pipe at rest, and the inviscid vibrational flow part, which considers all the perturbations relative to the mean flow resulting from the pipe vibration. Such assumptions are valid for sufficiently small mean-flow Mach numbers and sufficiently high vibrational-flow Reynolds numbers. Furthermore, the tube vibrational amplitude and the corresponding fluid vibrational velocity $v$ are assumed to be small enough so that the non-linear terms, such as $(v \cdot \nabla)v$, can be neglected.

Let $\rho$, $p$, and $v$ be primitive variables of the total flow in the measuring tube, standing for the fluid density, pressure and velocity, respectively, which can be split into:

$$\rho_t = \rho_0 + \rho, \quad p_t = P + p, \quad v_t = V + v,$$

where $\rho_0$, $P$ and $V$ represent the mean flow variables, and $\rho$, $p$ and $v$ the vibrational flow variables. The conservation equations for mass and momentum of the steady incompressible mean flow are:

$$\nabla \cdot V = 0, \quad \rho_0 \left( V \cdot \nabla \right) V = -\nabla P + \mu \nabla^2 V.$$

The vibrational flow equations are obtained by considering Eq. (5) into total flow equations and then subtracting the mean flow part (6). The conservation equations for mass and momentum of the low-amplitude inviscid vibrational flow can be written as:

$$\frac{1}{c^2} \left( \frac{\partial p}{\partial t} + V \cdot \nabla p \right) + \rho_0 \nabla \cdot v = 0, \quad \frac{\partial v}{\partial t} + \left( V \cdot \nabla \right) v + \left( v \cdot \nabla \right) V = -\frac{\nabla p}{\rho_0},$$

where the isentropic relationship between the vibrational pressure and the vibrational density, $p = c^2 \rho$, was taken into account. An identical system of equations was derived as the basis for the weight vector theory of the transit-time ultrasonic flowmeters by
Hemp (1982) and the Coriolis flowmeters by Hemp (1994) (the latter based on the assumption of incompressible flow).

Any further derivation of the mathematical model is limited to a straight cylindrical pipe with an axisymmetric, fully developed mean flow, which has the following velocity components in cylindrical polar coordinates \((x, r, \theta)\):

\[
\mathbf{V} = \left( V_z(r), 0, 0 \right). \tag{8}
\]

The pipe is assumed to vibrate periodic circumferentially and harmonic in time with the radial displacement:

\[
w_r(x, \theta, t) = \bar{w}_r(x) \cos n\theta \, e^{i\omega t}, \tag{9}
\]

which results in the fluid vibrational velocity field \(\mathbf{v} = (v_x, v_r, v_\theta)\):

\[
v_x(x, r, \theta, t) = \bar{v}_x(x, r) \cos n\theta \, e^{i\omega t}, \quad v_r(x, r, \theta, t) = \bar{v}_r(x, r) \cos n\theta \, e^{i\omega t},
\]

\[
v_\theta(x, r, \theta, t) = \bar{v}_\theta(x, r) \sin n\theta \, e^{i\omega t}, \tag{10}
\]

and the fluid vibrational pressure:

\[
p(x, r, \theta, t) = \bar{p}(x, r) \cos n\theta \, e^{i\omega t}. \tag{11}
\]

The dependency in coordinate \(x\) is resolved by the Fourier transform method. An integrable function \(f(x)\) and its Fourier transform \(f^*(\lambda)\) can be related as (Arfken and Weber, 2001):

\[
f^*(\lambda) = \int_{-\infty}^{\infty} f(x) e^{i\lambda x} \, dx, \tag{12}
\]

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(\lambda) e^{-i\lambda x} \, d\lambda, \tag{13}
\]

where \(\lambda\) is the axial wavenumber (Fourier variable). The Fourier transform of the derivative is \( (df/dx)^* = -i\lambda f^* \). Substituting Eqs. (8)–(11) into Eq. (7) and taking the Fourier transform in coordinate \(x\), we obtain:

\[
\frac{i}{c^2} (\omega - V_z \lambda) \bar{p}^* + \rho_0 \left( \frac{\partial \bar{v}_x^*}{\partial r} + \frac{\bar{v}_r^*}{r} + \frac{m\bar{v}_\theta^*}{r} - i\lambda \bar{p}^* \right) = 0, \tag{14}
\]

\[
i (\omega - V_z \lambda) \bar{v}_r^* + \bar{v}_r \frac{\partial \bar{v}_x^*}{\partial r} = \frac{i\lambda}{\rho_0} \bar{p}^*, \tag{15}
\]
\begin{equation}
    i(\omega - V_s \lambda) \vec{v}_r^* = -\frac{1}{\rho_0} \frac{\partial \vec{p}^*}{\partial r}, \quad (16)
\end{equation}

\begin{equation}
    i(\omega - V_s \lambda) \vec{v}_\theta^* = \frac{n}{\rho_0} \frac{\vec{p}^*}{r}. \quad (17)
\end{equation}

Next, the Fourier transforms of the vibrational velocities \( \vec{v}_r^* \), \( \vec{v}_\theta^* \) and \( \vec{v}_x^* \) are expressed with the Fourier transform of the vibrational pressure \( \vec{p}^* \) using Eqs. (15)–(17), and substituted into Eq. (14). The resulting differential equation for \( \vec{p}^* \) can be written as:

\begin{equation}
    \frac{\partial^2 \vec{p}^*}{\partial r^2} + \left( \frac{1}{r} + \frac{2\lambda}{\omega - V_s \lambda} \frac{dV_s}{dr} \right) \frac{\partial \vec{p}^*}{\partial r} - \left( \frac{n^2 + \lambda^2 - (\omega - V_s \lambda)^2}{r^2 - \frac{c^2}{\rho_0^2}} \right) \vec{p}^* = 0. \quad (18)
\end{equation}

Eq. (18) represents the so-called Pridmore-Brown (1958) equation. If the mean flow is uniform (\( \frac{d}{dr} V_s = 0 \)), this equation will be reduced to the Bessel-type differential equation. The non-uniform mean flow (\( \frac{d}{dr} V_s \neq 0 \)) is the source of an additional term near \( \frac{\partial \vec{p}^*}{\partial r} \). The Pridmore-Brown equation has many applications in studies of the hydrodynamic and acoustic modes of an inviscid fluid including the non-uniform mean-flow effects (see, e.g., Agarwal and Bull, 1989; Gogate and Munjal, 1993; Pagneux and Froelich, 2001; Willatzen, 2001; Boucher et al., 2006; Brambley et al., 2012). Only a few available references are focused on these effects with respect to the dynamics of a vibrating pipe (Pagneux and Auregan, 1998; Willatzen, 2003).

2.2. Boundary conditions

According to Myers (1980), the following boundary condition must be satisfied on the surface of the unperturbed pipe \( S_0 \) for the normal vibrational fluid velocity \( v_n \) with respect to the normal pipe displacement \( w_n \):

\begin{equation}
    v_n = \left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) w_n - w_n \n n \n \mathbf{V} \quad \text{at} \quad S_0. \quad (19)
\end{equation}

In our case \( v_n = v_r, \ n = w_r \) and \( S_0 \) is defined by \( r = R \). Because the mean flow has only the velocity component in the \( x \) direction, the last term in Eq. (19) vanishes and the boundary condition simplifies to:

\begin{equation}
    v_r = \left( \frac{\partial}{\partial t} + V_x \frac{\partial}{\partial x} \right) w_r \quad \text{at} \quad r = R. \quad (20)
\end{equation}

Considering Eqs. (9) and (10), and taking the Fourier transform, we obtain the boundary condition for \( \vec{v}_r^* \):
\[ \vec{v}_r = i(\omega - V_\lambda)x \vec{w}_r \quad \text{at} \quad r = R, \quad (21) \]

which is further expressed as the boundary condition for \( \vec{p}^* \) by using Eq. (16):

\[ \frac{\partial \vec{p}^*}{\partial r} = \rho_0 (\omega - V_\lambda)^2 \vec{w}_r \quad \text{at} \quad r = R. \quad (22) \]

In the case of a non-uniform mean flow with the non-slip condition \( V_\lambda(R) = 0 \), the boundary condition for \( \vec{p}^* \) becomes:

\[ \frac{\partial \vec{p}^*}{\partial r} = \rho_0 \omega^2 \vec{w}_r \quad \text{at} \quad r = R, \quad (23) \]

and in the case of the uniform mean flow \( V_\lambda(r) = \bar{V} \), the boundary condition for \( \vec{p}^* \) becomes:

\[ \frac{\partial \vec{p}^*}{\partial r} = \rho_0 (\omega - \bar{V})^2 \vec{w}_r \quad \text{at} \quad r = R. \quad (24) \]

Let us make the point that although the non-uniform mean flow does not have any component in the boundary condition (23), this does not mean no interaction of the mean flow with the vibrating tube – this interaction is taken into account within the calculation domain by the Pridmore-Brown equation (18). On the other hand, the interaction effects of the uniform mean flow are, almost completely, defined by the boundary condition (24).

2.3. Method of solution

The solution procedure for the Pridmore-Brown equation (18) will follow the Frobenius power series method (Arfken and Weber, 2001). Eq. (18) can be rewritten as

\[ \frac{\partial^2 \vec{p}^*}{\partial r^2} + q(r) \frac{\partial \vec{p}^*}{\partial r} + s(r) \vec{p}^* = 0, \]

and the Frobenius method assumes a solution in the form of a power series expansion in \( r \) around the regular singularities of \( q(r) \) and \( s(r) \). The possible singular points in the calculation domain of this equation are defined by \( r = 0 \) and \( \omega - V_\lambda(r) = 0 \). For the scope of this paper, the solution procedure can be reasonably limited to a singular point at \( r = 0 \). Namely, the second singularity, at the critical layer \( r = r_c \) defined by \( \omega - V_\lambda(r_c) = 0 \), occurs only for the modes with axial wavenumbers larger than \( \lambda_c = \omega/V_{\text{max}} \) (Shankar and Kumaran, 1999; Brambley et al., 2012). The corresponding solution has no contribution at small enough mean-flow velocities and does not affect the asymptotic model for the fluid-dynamic loading, which is derived in Section 3 by a power series expansion in the fluid velocity around
zero, up to the second order. The considered solution for $\tilde{P}^*$ being related to the $r = 0$ singularity covers the range $-\infty < \lambda < \lambda_{cr}$.

The Frobenius solution procedure in this paper is similar to that presented by Willatzen (2003) (but he started the solution from the linearized Pridmore-Brown equation, which would not lead to the correct centrifugal-acceleration components of the asymptotic model). Eq. (18) is multiplied by $r^2(\omega - V\tilde{\lambda})/\omega$ and $V_r(r)$ is substituted by the parabolic velocity profile (4) for the laminar mean flow. Defining next some dimensionless parameters:

$$\bar{r} = \frac{r}{R}, \quad \tilde{\lambda} = \lambda L, \quad \varepsilon_v = \frac{\nu}{\omega L}, \quad \varepsilon_L = \frac{R}{L}, \quad \varepsilon_c = \frac{\omega R}{c},$$

Eq. (18) can be written as:

$$\left(\sum_{j=0}^{2} a_j \bar{r}^j\right) \frac{\partial^2 \tilde{P}^*}{\partial \bar{r}^2} + \left(\sum_{j=0}^{2} b_j \bar{r}^j\right) \frac{\partial \tilde{P}^*}{\partial \bar{r}} + \left(\sum_{j=0}^{8} c_j \bar{r}^j\right) \tilde{P}^* = 0,$$

where the following coefficients $a_j$, $b_j$ and $c_j$ are generally not equal to zero:

$$a_0 = 1 - 2 \varepsilon_v \tilde{\lambda}, \quad a_2 = 2 \varepsilon_v \tilde{\lambda}, \quad b_0 = 1 - 2 \varepsilon_v \tilde{\lambda}, \quad b_2 = -6 \varepsilon_v \tilde{\lambda}, \quad c_0 = -n^2 \left(1 - 2 \varepsilon_v \tilde{\lambda}\right),$$

$$c_2 = -2n^2 \varepsilon_v \tilde{\lambda} - \left(1 - 2 \varepsilon_v \tilde{\lambda}\right) \varepsilon_c^2 + \left(1 - 2 \varepsilon_v \tilde{\lambda}\right)^3 \varepsilon_c^2, \quad c_4 = -2 \varepsilon_v \varepsilon_L^2 \tilde{\lambda}^3 + 6 \varepsilon_v \tilde{\lambda} \left(1 - 2 \varepsilon_v \tilde{\lambda}\right)^2 \varepsilon_c^2,$$

$$c_6 = 12 \varepsilon_v \varepsilon_L^2 \tilde{\lambda}^2 \left(1 - 2 \varepsilon_v \tilde{\lambda}\right) \varepsilon_c^2, \quad c_8 = 8 \varepsilon_v \tilde{\lambda}^3 \varepsilon_c^2.$$

With regard to the singularity at $\bar{r} = 0$, the method of Frobenius proposes a power series solution of the form:

$$\tilde{P}^* = \sum_{k=0}^{\infty} p_k \bar{r}^{k+k'} = p_0 \bar{r}^{k'} \left(1 + \sum_{k=1}^{\infty} \Pi_k \bar{r}^k\right),$$

with the exponent $k'$ and all the coefficients $p_k$ or $\Pi_k$ still undetermined. The proposed solution (28) is substituted into Eq. (26) and the coefficients of each power of $\bar{r}$ are combined and set to zero. The exponent $k'$ is determined by the indicial equation coming from the coefficients of the lowest power $\bar{r}^0$ and $k = 0$:

$$a_0 k' (k' - 1) + b_k k' + c_0 = 0,$$

which results in:

$$k' = \pm n.$$
Because the pressure $p^*$ has to remain finite inside the pipe, only $k' = n$ is physically realistic ($k' = -n$ leads to infinity of $p^*$ at $\bar{r} = 0$). The coefficients $\Pi_k$, $k \geq 1$, are determined by a recurrence relation coming from the coefficients of a higher power of $\bar{r}$:

$$
\Pi_k = -\sum_{j=1}^{k} \left[ a_j (k + n - j) (k + n - 1 - j) + b_j (k + n - j) + c_j \right] \Pi_{k-j} \over a_0 (k + n) (k + n - 1) + b_0 (k + n) + c_0 ,
$$

(31)

where $\Pi_{k-j} = 1$ for $k = j$. Finally, the coefficient $p_0$ is determined by employing the boundary condition (23) (at $\bar{r} = 1$):

$$
p_0 = \frac{R \rho \omega^2}{n + \sum_{k=1}^{\infty} (k + n) \Pi_k} \bar{w}_r^* ,
$$

(32)

and the Fourier transform of the vibrational pressure $p^*$ can be expressed as:

$$
p^* = R \rho \omega^2 \frac{R \rho \omega^2}{n + \sum_{k=1}^{\infty} (k + n) \Pi_k} \bar{w}_r^* \).
$$

(33)

Next, taking the inverse Fourier transform of $p^*$ (Eq. (13)) and substituting the expression for $\bar{w}_r^*$ (Eq. (12), $\bar{w}_r^* (x) = 0$ for $x < 0$ and $x > L$), we obtain the fluid-dynamic pressure on the pipe surface (at $\bar{r} = 1$):

$$
\bar{p}_d (x) = R \rho \omega^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 + \sum_{k=1}^{\infty} \Pi_k}{n + \sum_{k=1}^{\infty} (k + n) \Pi_k} \int_{0}^{L} \bar{w}_r^* (y) e^{i(x-y)} dy \, d\lambda .
$$

(34)

The Frobenius method of solution can also be similarly employed for a pipe with the uniform mean flow. Such a solution, as expected, converges to the results of Eq. (2).

In the case of a beam-type pipe with $n = 1$, the fluid-dynamic loading (34) can be expressed as the force per unit length

$$
f_d (x, t) = f_d (x) e^{i\omega t} .
$$
11

which is related to the pipe lateral deflection \( w(x,t) = \overline{w}(x)e^{i\omega t} \).

3. Asymptotic model for the fluid-dynamic loading

As shown in Section 1, the fluid-dynamic loading of pipes conveying fluid with a uniform mean-flow velocity profile consists of three components up to the second order in the fluid velocity, which can be related to the translational, Coriolis and centrifugal accelerations. In general, the non-uniform mean flow gives rise to additional higher-order components. The asymptotic model for the fluid-dynamic loading presented in this paper is limited to the components up to the second order in the fluid velocity. The derivation is realized by a power series expansion in \( \varepsilon_r = \frac{\partial}{\partial L} \) around \( \varepsilon_r = 0 \) up to order \( \varepsilon_r^2 \), which is valid for a relatively small fluid velocity, with the remainder \( \Re(\varepsilon_r^4) \ll 1 \). Furthermore, the asymptotic model is simplified with respect to the effects of the pipe length and the fluid compressibility, by performing additional power series expansions in \( \varepsilon_L = \frac{R}{L} \) around \( \varepsilon_L = 0 \) up to order \( \varepsilon_L^2 \) and in \( \varepsilon_c = \frac{\omega R}{c} \) around \( \varepsilon_c = 0 \) up to order \( \varepsilon_c^2 \), respectively. Such an asymptotic model is valid for a relatively long pipe, \( \Re(\varepsilon_L^4) \ll 1 \), and for a relatively small fluid compressibility, \( \Re(\varepsilon_c^4) \ll 1 \), and we also neglect the mixed terms that have a similar order of magnitude, \( \Re(\varepsilon_L^2\varepsilon_c^2) \ll 1 \).

Utilizing the given procedure, the asymptotic model for the fluid-dynamic pressure \( \overline{p}_d(x) \) can be expressed as:

\[
\overline{p}_d(x) = \frac{\rho_0 R}{n} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \alpha_0 \omega^2 - \alpha_1 2\Gamma \omega \lambda + \alpha_2 \varepsilon^2 \lambda^2 \right) \int_0^L \overline{w}(y) e^{i\lambda(y-x)} dy d\lambda , \tag{36}
\]

where the coefficients \( \alpha_j \) refer to the particular velocity terms with power \( \vartheta^j / \lambda^j \):

\[
\alpha_j(\lambda) = \alpha_j^{(0)} \left[ 1 - \alpha_j^{(1)} \frac{\rho R}{c} \lambda^2 + \alpha_j^{(2)} \left( \frac{\rho R}{c} \right)^2 \right], \quad \text{for } j = 0, 1, 2. \tag{37}
\]

Considering the identity for the Fourier integral:
we obtain the following asymptotic model for the fluid-dynamic pressure \( p_d(x, \theta, t) \) = \( \bar{p}_d(x) \cos n\theta e^{i\omega t} \) being related to the pipe radial displacement \( w_r(x, \theta, t) \):

\[
p_d(x, \theta, t) = -\frac{\rho_0 R}{n} \left( \alpha_0 \frac{\partial^2 w_r}{\partial t^2} + \alpha_1 \frac{\partial^2 w_r}{\partial t \partial x} + \alpha_2 \frac{\partial^2 w_r}{\partial x^2} \right),
\]

where

\[
\alpha_j = \alpha_j^{(0)} \left[ 1 + \alpha_j^{(e_x)} R^2 \left( \frac{\omega R}{c} \right)^2 + \alpha_j^{(e_z)} \left( \frac{\omega R}{c} \right)^2 \right], \quad \text{for } j = 0, 1, 2.
\]

In the case of the beam-type pipe with \( n = 1 \), the fluid-dynamic loading can be expressed as the force per unit length:

\[
f_d(x, t) = -M_f \left( \alpha_0 \frac{\partial^2 w}{\partial t^2} + \alpha_1 \frac{\partial^2 w}{\partial t \partial x} + \alpha_2 \frac{\partial^2 w}{\partial x^2} \right).
\]
### Table 1 Basic correction factors, $\alpha^{(0)}$.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Uniform profile</strong></td>
<td>$V_x = \overline{V}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td><strong>Laminar profile</strong></td>
<td>$V_x = 2\overline{V} \left(1 - \frac{r^2}{R^2}\right)$</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

|                     | $n=1 \rightarrow 0.25$ | $n=1 \rightarrow 0.25$ | $n=1 \rightarrow 0.25$ |
|                     | $n=2 \rightarrow 0.083$ | $n=2 \rightarrow 0.083$ | $n=2 \rightarrow 0.083$ |
|                     | $n=3 \rightarrow 0.042$ | $n=3 \rightarrow 0.042$ | $n=3 \rightarrow 0.042$ |

### Table 2 Correction factors for the pipe-length effect, $\alpha^{(L)}$.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Uniform profile</strong></td>
<td>$V_x = \overline{V}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td><strong>Laminar profile</strong></td>
<td>$V_x = 2\overline{V} \left(1 - \frac{r^2}{R^2}\right)$</td>
<td>(2n(n+1))</td>
<td>(2n(n+1))</td>
</tr>
</tbody>
</table>

|                     | $n=1 \rightarrow 0.25$ | $n=1 \rightarrow 0.25$ | $n=1 \rightarrow 0.25$ |
|                     | $n=2 \rightarrow 0.083$ | $n=2 \rightarrow 0.083$ | $n=2 \rightarrow 0.083$ |
|                     | $n=3 \rightarrow 0.042$ | $n=3 \rightarrow 0.042$ | $n=3 \rightarrow 0.042$ |

|                     | \(n=1 \rightarrow 0.417\) | \(n=1 \rightarrow 0.417\) | \(n=1 \rightarrow 0.938\) |
|                     | \(n=2 \rightarrow 0.146\) | \(n=2 \rightarrow 0.146\) | \(n=2 \rightarrow 0.35\) |
|                     | \(n=3 \rightarrow 0.075\) | \(n=3 \rightarrow 0.075\) | \(n=3 \rightarrow 0.188\) |

### Table 3 Correction factors for the fluid-compressibility effect, $\alpha^{(C)}$.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Uniform profile</strong></td>
<td>$V_x = \overline{V}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td><strong>Laminar profile</strong></td>
<td>$V_x = 2\overline{V} \left(1 - \frac{r^2}{R^2}\right)$</td>
<td>(2n(n+1))</td>
<td>(n(n+1))</td>
</tr>
</tbody>
</table>

|                     | $n=1 \rightarrow 0.25$ | $n=1 \rightarrow 0.5$ | $n=1 \rightarrow 1.5$ |
|                     | $n=2 \rightarrow 0.083$ | $n=2 \rightarrow 0.167$ | $n=2 \rightarrow 0.5$ |
|                     | $n=3 \rightarrow 0.042$ | $n=3 \rightarrow 0.083$ | $n=3 \rightarrow 0.25$ |

|                     | \(n=1 \rightarrow 0.583\) | \(n=1 \rightarrow 0.583\) | \(n=1 \rightarrow 2.688\) |
|                     | \(n=2 \rightarrow 0.208\) | \(n=2 \rightarrow 0.208\) | \(n=2 \rightarrow 1.3\) |
|                     | \(n=3 \rightarrow 0.108\) | \(n=3 \rightarrow 0.108\) | \(n=3 \rightarrow 0.854\) |
3.1. Translational-acceleration component

The translational-acceleration component of the fluid-dynamic loading does not depend on the mean-flow velocity, so it is also not affected by the velocity distribution. The relative compressibility effect is \( \alpha_0^{(C)} (\omega R / c)^2 \) with \( \alpha_0^{(C)} = 0.25 \) for the beam-type mode \((n = 1)\), which equals the compressibility effect on the fluid-density measurement in Coriolis flow meters, theoretically predicted by Hemp and Kutin (2006). Such an agreement is reasonable because Coriolis flow meters determine the fluid density from its effect on the translational inertia and consequently on the natural frequency of the pipe. Both the fluid-compressibility and the pipe-length effects decrease with the circumferential wavenumber \( n \) (which holds true for all the components of the fluid-dynamic loading).

3.2. Coriolis-acceleration component

In the case of relatively long pipes conveying an incompressible fluid, the Coriolis-acceleration component of the fluid-dynamic loading is found to be independent of the mean-flow velocity profile for the beam-type mode \((\alpha_0^{(b)} = 1 \text{ for } n = 1)\), whereas the shell-type modes \((n > 1)\) lead to its decrease by a factor \( \alpha_2^{(b)} = 2 / (n + 1) \). For shorter tubes with non-negligible pipe-length effects, the velocity profile effects also arise for the beam-type mode (different \( \alpha_i^{(b)} \) for the uniform and the laminar velocity profiles). These results are in perfect agreement with the correction factors for the Coriolis flow measuring effects, which were estimated analytically by Kutin at al. (2005) using the weight vector theory. The confidence in the given analytical results is also supported by the fact that the principal findings agree with the available numerical and experimental studies of the velocity profile effects in Coriolis flow meters (see, e.g., Grimley, 2002; Bobovnik at al., 2004, 2005; Mole at al., 2008; Kumar and Anklin, 2011).

For the beam-type mode \((n = 1)\), the relative compressibility effect on the Coriolis-acceleration component of the fluid-dynamic loading is \( \alpha_1^{(C)} (\omega R / c)^2 \) with \( \alpha_1^{(C)} = 0.5 \) for the uniform velocity profile and \( \alpha_1^{(C)} = 0.583 \) for the laminar velocity profile. Identical theoretical results were found by Hemp and Kutin (2006) for the compressibility effect on the flow measurement in Coriolis flow meters. These analytical results are in good agreement with the experimental observations of Anklin et al. (2000), where the correction factor \( \alpha_1^{(C)} \) was found to be between 0.496 and 0.608.

3.3. Centrifugal-acceleration component

In the case of relatively long pipes conveying an incompressible fluid and vibrating in the beam-type mode \((n = 1)\), the laminar velocity profile is found to decrease the centrifugal-acceleration term of the fluid-dynamic loading by a factor \( \alpha_2^{(b)} = 2 / 3 \). This result is different from the correction factor of \( 4 / 3 \) proposed by Guo et al. (2010) and Hellum et al. (2010) following a definition of the momentum correction.
factor. With the intention of explaining such a discrepancy, we take a look at the physical background of the fluid-structure interaction in the system under discussion. The transverse vibrational velocity field of a fluid with a uniform mean-flow velocity profile can be written as:

\[ v_y = \frac{\partial w}{\partial t} + \bar{V} \frac{\partial w}{\partial x}, \quad v_z = 0, \quad (42) \]

where \( v_y \) represents the velocity component in the direction of the pipe vibration. The approach used by Guo et al. (2010) and Hellum et al. (2010) would be correct only if the non-uniform mean flow also entirely followed the pipe vibration; thus the transverse vibrational velocity field of the fluid with the laminar mean-flow velocity profile would be:

\[ v_y^{(\text{lam})} = \frac{\partial w}{\partial t} + 2 \left( 1 - \frac{r^2}{R^2} \right) \bar{V} \frac{\partial w}{\partial x}, \quad v_z^{(\text{lam})} = 0. \quad (43) \]

However, the mathematical model presented in this paper gives the following transverse vibrational field in the case of a laminar mean flow (approximate solution up to the first order in the fluid velocity):

\[ v_y^{(\text{lam})} = \frac{\partial w}{\partial t} + \left( 1 - \frac{r^2}{R^2} \cos 2\theta \right) \bar{V} \frac{\partial w}{\partial x}, \quad v_z^{(\text{lam})} = -\frac{r^2}{R^2} \sin 2\theta \frac{\partial w}{\partial x}. \quad (44) \]

The difference between Eqs. (44) and (43) represents the transverse secondary-flow field:

\[ \Delta v_y^{(\text{lam})} = \left[ -1 + \frac{r^2}{R^2} (2 - \cos 2\theta) \right] \bar{V} \frac{\partial w}{\partial x}, \quad \Delta v_z^{(\text{lam})} = -\frac{r^2}{R^2} \sin 2\theta \bar{V} \frac{\partial w}{\partial x}. \quad (45) \]

As shown in the vector plot in Fig. 2 (for \( \partial w / \partial x > 0 \)), the secondary flow consists of a pair of counter-rotating vortices. Similar secondary-flow patterns can be found in the literature studying the viscous flows through curved pipes (see, e.g., Berger and Talbot, 1983; Gammack and Hydon, 2001). Thus, the secondary flow in a vibrating pipe with a laminar mean flow is not something unexpected. On the other hand, Guo et al. (2010) and Hellum et al. (2010) omitted the effects of the secondary flow, so we think that the validity of their proposed correction factors for the centrifugal-acceleration component is questionable.
4. Application case – a beam-type pipe conveying fluid, with both ends clamped

This application case for the discussed mathematical model of the fluid-dynamic loading assumes a pipe with flexural rigidity $EI$ and mass per unit length $M_p$, which vibrates in the bending, beam-type mode ($n = 1$) and is modelled in terms of the Euler-Bernoulli beam theory. This beam theory neglects the effects of rotary inertia and shear deformation and is thus applicable to an analysis of slender pipes (see, e.g., Kutin and Bajsić (1999) for some comparison results between the Euler-Bernoulli beam theory and the more accurate Flügge shell theory). Neglecting the vibration damping, the equation of motion can be written as:

$$EI \frac{\partial^4 w}{\partial x^4} + M_p \frac{\partial^2 w}{\partial t^2} = f_d(x,t),$$

where the first two terms characterize the dynamics of the Euler-Bernoulli beam and $f_d(x,t)$ is the fluid-dynamic force per unit length related to the pipe deflection $w(x,t)$, discussed in previous sections. The pipe is assumed to have clamped boundary conditions at both ends:

$$w = \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0, \ x = L.$$

Considering a time-harmonic vibration of the angular frequency $\omega$, $w(x,t) = \overline{w}(x)e^{i\omega t}$ and $f_d(x,t) = \overline{f}_d(x)e^{i\omega t}$, Eqs. (46) and (47) take the form:

$$EI \frac{d^4 \overline{w}}{dx^4} - \omega^2 M_p \overline{w} = \overline{f}_d(x),$$

$$\overline{w} = \frac{d\overline{w}}{dx} = 0 \quad \text{at} \quad x = 0, \ x = L.$$
With respect to generalization, we next define some dimensionless parameters using the reduction on the tube parameters $L$, $M_p$, and $EI$:

$$
\tilde{x} = \frac{x}{L}, \quad \tilde{\eta} = \frac{\eta}{L}, \quad \tilde{\psi}_d = \frac{L^5}{EI} \int_0^1 \tilde{f}_d, \quad \beta = \frac{M_f}{M_p}, \quad \nu = \frac{M_p}{EI}, \quad \Omega = \omega L^2 \sqrt{\frac{M_p}{EI}}.
$$

(50)

Thus, Eqs. (48) and (49) can be written as:

$$
\frac{d^4 \tilde{\eta}}{dx^4} - \Omega^2 \tilde{\eta} = \tilde{\psi}_d (\tilde{x}),
$$

(51)

$$
\tilde{\eta} = \frac{d \tilde{\eta}}{dx} = 0 \quad \text{at} \quad \tilde{x} = 0, \quad \tilde{x} = 1,
$$

(52)

where the dimensionless amplitude of the fluid-dynamic loading can be expressed as (from Eq. (35)):

$$
\tilde{\psi}_d (\tilde{x}) = \beta \Omega^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} \Pi_k \left( \int_0^1 \tilde{\eta} (\tilde{y}) e^{i(\lambda_k - \lambda_m) \tilde{x}} d\tilde{y} \right) d\tilde{x},
$$

(53)

and its asymptotic approximation can be expressed as (from Eq. (41)):

$$
\tilde{\psi}_d (\tilde{x}) = \alpha_n \beta \Omega^2 \tilde{\eta} - \alpha_2 \nu \Omega \frac{d \tilde{\eta}}{dx} - \alpha_2 \nu \beta^2 \frac{d^2 \tilde{\eta}}{dx^2}.
$$

(54)

Using the Galerkin solution procedure, the general solution for the equation of motion (51) can be given as:

$$
\tilde{\eta} (\tilde{x}) = \sum_{m=1}^{\infty} C_m \phi_m (\tilde{x}),
$$

(55)

where $\phi_m (\tilde{x})$ are a complete set of functions that identically satisfy the boundary conditions. Let us define $\phi_m (\tilde{x})$ using the eigenfunctions of the Euler-Bernoulli beam with clamped boundary conditions (52):

$$
\phi_m (\tilde{x}) = \cosh \lambda_m \tilde{x} - \cos \lambda_m \tilde{x} - \cosh \lambda_m - \cos \lambda_m (\sinh \lambda_m \tilde{x} - \sin \lambda_m \tilde{x}),
$$

(56)

where the eigenvalues $\lambda_m$ follow from the equation:
\[ \cos \lambda \cosh \lambda = 1, \text{ i.e. } \lambda_1 = 4.730, \lambda_2 = 7.853, \ldots, \lambda_m \approx \left( m + \frac{1}{2} \right) \pi. \] (57)

Following the Galerkin solution procedure, Eq. (55) is substituted into Eq. (51), the resulting equation is multiplied by \( \varphi_l(\hat{x}) \), \( l \geq 1 \), and integrated with respect to \( \hat{x} \) from 0 to 1. The procedure leads to a homogeneous set of linear equations in the constants \( C_m \). For non-trivial solutions, the determinant of the coefficients has to vanish, thus yielding a transcendental equation in the dimensionless frequency \( \Omega \). Its positive real parts represent the natural frequencies of the pipe for different axial modes.

Fig. 3 illustrates the effect of the mean-flow velocity profiles on the first-mode natural frequency for a relatively long pipe conveying an incompressible fluid \( (\Re \ll \varepsilon, \C \ll \varepsilon) \). The laminar profile leads to a smaller flow effect on the natural frequency in comparison with the uniform profile, which is mainly related to the smaller correction factor for the centrifugal-acceleration component of the fluid-dynamic loading (see the asymptotic model for the fluid-dynamic loading that has \( \alpha_2^{(0)} = 2/3 \) for the laminar profile and \( \alpha_2^{(0)} = 1 \) for the uniform profile). For the uniform velocity profile, the results of the asymptotic model are equal to the complete-model solutions, so they are not separately presented in Fig. 3. The observed deviations of the asymptotic model for the laminar velocity profile are partially related to neglected higher-order velocity components, which are generally present in the case of non-uniform mean flows. Furthermore, the part of the solution of the Pridmore-Brown equation that corresponds to the critical-layer singularity, and was not taken into account in this paper, may have some contribution for higher fluid velocities in the range of about \( \nu > 0.8 \).

Next, Figs. 4 and 5 illustrate the pipe-length and the fluid-compressibility effects on the first-mode natural frequency, respectively. In this case the asymptotic model for the pipe-length effect shows good agreement up to pipe-length ratios of about \( \varepsilon_L = R / L = 0.05 \), i.e., for the pipes with a length-to-diameter ratio higher than 10. The asymptotic model for the fluid-compressibility effect shows good agreement up to fluid-compressibility ratios of about \( \varepsilon_c = \omega R / c = 0.25 \).
Fig. 3. Variation of the dimensionless first-mode natural frequency with the dimensionless fluid velocity for the uniform and laminar velocity profiles ($\beta = 1, \Re(e_L^2) \ll 1, \Re(e_C^2) \ll 1$).

Fig. 4. Variation of the dimensionless first-mode natural frequency with the pipe-length ratio for the uniform and laminar velocity profiles ($\beta = 1, \nu = 0.5, \Re(e_C^2) \ll 1$).
Fig. 5. Variation of the dimensionless first-mode natural frequency with the fluid-compressibility ratio for the uniform and laminar velocity profiles ($\beta = 1, \nu = 0.5, \Re(\varepsilon_L^2) \ll 1$).

All the calculations in this section were performed with the Galerkin series truncated at $m = 10$ ($l = 10$) and the Frobenius series truncated at $k = 20$. With a view to checking the convergence level of the solutions, we performed a few additional calculations with the number of terms increased to $m = 15$ ($l = 15$) and $k = 30$, respectively. Such an increase in the number of terms in the Galerkin series does not have noticeable influence on the calculated natural frequencies, with the changes being less than 0.01%. A similar conclusion is found for the increased number of terms in the Frobenious series; an exception is the full-model solution in Fig. 3 for the laminar profile at higher fluid velocities, where the changes reach order 0.1%.

5. Conclusions

The aim of this paper was to present the effects of the mean flow with a laminar velocity profile on the fluid-dynamic loading of a straight pipe conveying fluid. The main contribution is the derived asymptotic model for the fluid-dynamic loading, which considers the flow effects up to the second order in the fluid velocity and is applicable for any circumferential wavenumber of the mode shape of a pipe. This model also takes into account some range of the effects of the fluid compressibility and the finite pipe length. The asymptotic model consists of three components, which can be related to the translational, Coriolis and centrifugal accelerations of the fluid. The velocity profile effects are expressed in terms of the corresponding correction factors that were calculated for the laminar and uniform velocity distributions.
There are not many accessible studies that deal with the velocity profile effects on the fluid-dynamic loading of pipes conveying fluid. Nevertheless, these effects have been extensively studied in the field of Coriolis flow metering. The velocity profile effects on the flow measurement in these meters can be directly related to the correction factors for the Coriolis-acceleration component of the asymptotic model. It is important for the confidence in the derived asymptotic model that its results agree with the analytical, numerical and experimental findings on the velocity profile effects in Coriolis flow meters. On the other hand, a major discrepancy was found for the basic correction factor for the centrifugal-acceleration component when compared to the work of Guo et al. (2010) and Hellum et al. (2010). We assessed its value to be 2/3 for the laminar flow in a beam-type pipe, but they, following a definition of the momentum correction factor, found 4/3 for the same conditions. The analysis of the physical background in the present paper reveals a doubt about the validity of their result, because they omitted the effects of the secondary flow that arises for a non-uniform mean flow in a vibrating pipe. The different value of the correction factor for the centrifugal-acceleration component (compared with unity for the uniform flow) entails that Guo et al. (2010) and Hellum et al. (2010) predicted dynamic instabilities (flutter) of cantilevered beam-type pipes at lower critical velocities for the laminar flow, while the model of this paper would predict higher critical velocities for the laminar flow.

The derived asymptotic model for the fluid-dynamic loading of the laminar mean flow is valid for sufficiently small fluid velocities. For the evaluation of its validity at higher fluid velocities a solution of the Pridmore-Brown equation will certainly need to be complemented with the part corresponding to the critical-layer singularity. Nevertheless, an important question remains about the suitability of all these models for predictions of the particular instabilities of pipes conveying fluid. The clamped-clamped pipes, for instance, typically lose their stability due to static buckling or divergence, which occurs at the zero frequency (Païdoussis, 1998, 2004). But the employed wave models are based on the assumption of an inviscid vibrational flow, which is valid for sufficiently high vibrational-flow Reynolds numbers, and accordingly for sufficiently high vibration frequencies. It might be an idea to search approximate models for the pipe loading under static instabilities in the framework of the analytical mathematical models for fluid flow through curved or wavy pipes at rest (see, e.g., Murata et al., 1976; Berger and Talbot, 1983; Gammack and Hydon, 2001; Khuri, 2006).

References


